TUGAS KELAS
PEMBAHASAN SOAL CHEBYSHEV

Disusun Untuk Memenuhi Salah Satu Tugas Mata Kuliah
PENGANTAR TEORI PELUANG

Dosen :
Ir. Agus Purwoto, M.Si

Disusun Oleh :

KELAS 1 – D

SEKOLAH TINGGI ILMU STATISTIK
JAKARTA
2011

1. Bentuk ketidaksamaan Chebyshev, pada :

a) **Distribusi Normal** $X \approx N(\mu ; \sigma^2)$
$\mu = \mu$
$\sigma^2 = \sigma^2$

$P\,[|x - \mu| \;\geq\; r\sigma] \;\leq\; \frac{1}{r^2}$ □ Batas Atas

$P\,[|x - \mu| \;\leq\; r\sigma] \;\geq\; 1 - \frac{1}{r^2}$ □ Batas Bawah

b) **Distribusi Normal Baku** $X \approx N(0,1)$
$\mu = 0$
$\sigma^2 = 1$

$P\,[|x| \;\geq\; r] \;\leq\; \frac{1}{r^2}$ □ Batas Atas

$P\,[|x| \;\leq\; r] \;\geq\; 1 - \frac{1}{r^2}$ □ Batas Bawah

c) **Distribusi Poisson** $X \approx N(\lambda ; \lambda)$
$\mu = \lambda$
$\sigma^2 = \lambda$

$P\,[|x - \lambda| \;\geq\; r\sqrt{\lambda}] \;\leq\; \frac{1}{r^2}$ □ Batas Atas

$P\,[|x - \lambda| \;\leq\; r\sqrt{\lambda}] \;\geq\; 1 - \frac{1}{r^2}$ □ Batas Bawah

d) **Distribusi Binomial** $X \approx N(np ; npq)$
$\mu = np$
$\sigma^2 = npq$

$P\,[|x - np| \;\geq\; r\sqrt{npq}\,] \;\leq\; \frac{1}{r^2}$ □ Batas Atas

$P\,[|x - np| \;\leq\; r\sqrt{npq}\,] \;\geq\; 1 - \frac{1}{r^2}$ □ Batas Bawah

2. $P[|x| \geq k] \leq \frac{1}{k^2}$ jika x mengikuti N (0,1)

Diketahui :

$\mu = 0$

$\sigma^2 = 1$

$\sigma = 1$

misalkan $k = r\sigma$

$k = r$

Solusi :

$$P(|x-\mu| \geq r\sigma) \leq \frac{1}{r^2}$$

Dengan mensubstitusikan $\mu = 0$ dan $\sigma = 1$, maka diperoleh

$$P[|x-0| \geq r(1)] \leq \frac{1}{r^2}$$

$$P[|x| \geq k] \leq \frac{1}{k^2} \text{ (terbukti)}$$

3. Suatu fungsi diketahui :

$$f(x) = \frac{1}{2^x} \quad , x = 1, 2, 3, \ldots$$

$$= 0, \text{untuk x yang lainnya}$$

Buktikan :

$P[|x-\mu| \geq 2] \leq \frac{1}{2}$ sedangkan peluang sesungguhnya bernilai $\frac{1}{16}$

Sebelum membuktikan peluang Chebyshev, kita perlu mencari nilai μ dan σ^2

Pencarian nilai μ menggunakan MGF (*Moment Generating Function*)

$$M_x(t) = E(e^{tx}) = \sum_{k=1}^{\infty} e^{tx} f(x)$$

$$M_x(t) = \sum_{k=1}^{\infty} e^{tx} \frac{1}{2^x}$$

$$M_x(t) = \frac{e^t}{2} + \frac{e^{2t}}{2^2} + \frac{e^{3t}}{2^3} + \ldots$$

$$M_x(t) = \frac{\frac{e^t}{2}}{1 - \frac{e^t}{2}} = \frac{\frac{e^t}{2}}{\frac{2-e^t}{2}} = \frac{e^t}{2-e^t}$$

$$M_x{}'(t) = \frac{e^t(2-e^t) - (-e^t)(e^t)}{(2-e^t)^2}$$

$$M_x{}'(t) = \frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2} = \frac{2e^t}{(2-e^t)^2}$$

$$\mu = E(X) = M_x'(t=0) = \frac{2.e^0}{(2-e^0)^2} = 2$$

$$M_x''(t) = E(X^2)$$

$$M_x{}' = \frac{2e^t}{(2-e^t)^2}$$

$$M_x''(t) \quad = \quad \frac{2e^t(2-e^t)^2 \ + \ 2(2-e^t)(e^t)(2e^t)}{(2-e^t)^4}$$

$$M_x''(t=0) \quad = \quad \frac{2e^0(2-e^0)^2 \ + \ 2(2-e^0)(e^0)(2e^0)}{(2-e^0)^4}$$

$$M_x''(t=0) \quad = \quad \frac{2(1)(2-1)^2 \ + \ 2(2-1)(1)(2(1))}{(2-1)^4}$$

$$M_x''(t=0) = E(X^2) = \frac{2+4}{1} = 6$$

$$\sigma^2 \ = \ E(X^2) - (E(X))^2$$

$$\sigma^2 \ = \ 6 - (2)^2 = 2$$

Dari hasil pencarian di atas didapatkan nilai

$\mu = 2$

$\sigma^2 = 2$

$\sigma = \sqrt{2}$

$r\sigma = 2$

$r = \frac{2}{\sqrt{2}} = \sqrt{2}$

$P[|x-\mu| \geq r\sigma] \leq \frac{1}{r^2}$

$P[|x-2| \geq \sqrt{2}(\sqrt{2})] \leq \frac{1}{(\sqrt{2})^2}$

$P[|x-2| \geq 2] \leq \frac{1}{2}$

$P[|x-\mu| \geq 2] \leq \frac{1}{2}$ (TERBUKTI)

Dengan peluang sesungguhnya

$P[|x-\mu| \geq 2] = P[|x-2| \geq 2]$

$P[|x-2| \geq 2] = 1 - P[|x-2| \leq 2]$

$P[|x-2| \geq 2] = 1 - P[0 \leq x \leq 4] = 1 - P[1 \leq x \leq 4]$ (Karena x bergerak dari 1)

$P[|x-2| \geq 2] = 1 - \left(\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right)$

$P[|x-2| \geq 2] = 1 - \frac{15}{16} = \frac{1}{16}$ (TERBUKTI)

4. Dik:

$\mu = 12$

$\sigma^2 = 4$

$\sigma = 2$

a. P $[6 < x < 18]$

$P\,[|x - 12| < r\sigma] \geq 1 - \frac{1}{r^2}$

$P\,[|x - 12| < 6] \geq 1 - \frac{1}{r^2}$

$r\sigma = 6$

$2r = 6$

$r = 3$

$P\,[|x - 12| < 6] \geq 1 - \frac{1}{3^2}$

$P\,[|x - 12| < 6] \geq \frac{8}{9}$

$P\,[6 < x < 18] \geq \frac{8}{9}$

b. P $(3 < x < 21)$

$P\,[|x - 12| < r\sigma] \geq 1 - \frac{1}{r^2}$

$P\,[|x - 12| < 9] \geq 1 - \frac{1}{r^2}$

$r\sigma = 9$

$2r = 9$

$r = \frac{9}{2}$

$P\,[|x - 12| < 9] \geq 1 - \frac{1}{\left(\frac{9}{2}\right)^2}$

$P\,[|x - 12| < 9] \geq \frac{77}{81}$

$P\,[3 < x < 21] \geq \frac{77}{81}$

5. Dik :

$\mu = 10$

$\sigma^2 = 9$

$\sigma = 3$

a. $P\,[\,|\,x - 10\,| \geq 3\,]$

$3 = r\sigma \quad r = \frac{3}{\sigma} = \frac{3}{3} = 1$

$P\,[\,|\,x - 10\,| \geq 3\,] \leq \frac{1}{r^2}$

$P\,[\,|\,x - 10\,| \geq 3\,] \leq 1$

b. $P[\,|x - 10| < 3\,]$

Dari pernyataan diatas didapat r = 1

$$P[\,|x - 10| < 3\,] \geq 1 - \frac{1}{1^2}$$

$$P[\,|x - 10| < 3\,] \geq 0$$

c. $P\,[5 < x < 15]$

$$P\,[5 < x < 15\,] = P[\,|x - 10| < 5\,]$$

$$5 = r\sigma$$

$$5 = 3r$$

$$r = \frac{5}{3}$$

$$P[\,|x - 10| < 5\,] \geq 1 - \frac{1}{r^2}$$

$$P[\,|x - 10| < 5\,] \geq 1 - \frac{1}{\left(\frac{25}{9}\right)}$$

$$P[\,|x - 10| < 5\,] \geq \frac{16}{25}$$

$$P\,[5 < x < 15] \geq \frac{16}{25}$$

d. Nilai c

$$P[\,|x - 10| \geq c\,] \leq 0{,}04$$

$$P[\,|x - 10| \geq c\,] \leq 0{,}04$$

$$P[\,|x - 10| \geq c\,] \leq \frac{1}{r^2}$$

$$\frac{1}{r^2} = 0{,}04 = \frac{4}{100}$$

$$r^2 = \frac{100}{4} = 25$$

$$r = \sqrt{\frac{100}{4}} = \frac{10}{2} = 5$$

$$c = r\sigma = 5(3) = 15$$

6. Dik :

$$f(x) = 6x(1 - x), \quad 0 < x < 1$$
$$= 0\,, \quad \text{ow}$$

Hitung $P[\,|x - \mu| < 2\sigma]$

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)dx$$

$$E(X) = \int_{0}^{1} x(6x)(1 - x)dx$$

$$E(X) = \int_{0}^{1} (6x^2 - 6x^3)\, dx$$

$$E(X) = \left(2x^3 - \frac{3}{2}x^4\right)\Big|_0^1$$

$$E(X) = \mu = 2 - \frac{3}{2} = \frac{1}{2}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(X^2) = \int_0^1 x^2\ (6x)(1-x)dx$$

$$E(X^2) = \int_0^1 6x^3 - 6x^4\ \ dx$$

$$E(X^2) = \left(\frac{3}{2}x^4 - \frac{6}{5}x^5\right)\Big|_0^1$$

$$E(X^2) = \frac{3}{2} - \frac{6}{5}$$

$$E(X^2) = \frac{3}{10}$$

$$\sigma^2 = E(X^2) - (E(X))^2$$

$$\sigma^2 = \frac{3}{10} - \left(\frac{1}{2}\right)^2$$

$$\sigma^2 = \frac{1}{20}$$

$$\sigma = \frac{1}{2\sqrt{5}}$$

Cara Chebyshev :

$$P[\,|x - \mu| < 2\sigma] \geq 1 - \frac{1}{r^2}$$

$$P\left[\left|x - \frac{1}{2}\right| < 2\left(\frac{1}{2\sqrt{5}}\right)\right] \geq 1 - \frac{1}{2^2}$$

$$P\left[\left|x - \frac{1}{2}\right| < \frac{1}{\sqrt{5}}\right] \geq \frac{3}{4}$$

Peluang sesungguhnya :

$$P\left[\left(-\frac{1}{\sqrt{5}} + \frac{1}{2}\right) < x < \left(\frac{1}{\sqrt{5}} + \frac{1}{2}\right)\right]$$

$$P\left[\left(-\frac{1}{\sqrt{5}} + \frac{1}{2}\right) < x < \left(\frac{1}{\sqrt{5}} + \frac{1}{2}\right)\right] = \int_{-\frac{1}{\sqrt{5}}+\frac{1}{2}}^{\frac{1}{\sqrt{5}}+\frac{1}{2}} 6x(1-x)dx$$

$$P\left[\left(-\frac{1}{\sqrt{5}} + \frac{1}{2}\right) < x < \left(\frac{1}{\sqrt{5}} + \frac{1}{2}\right)\right] = \int_{-\frac{1}{\sqrt{5}}+\frac{1}{2}}^{\frac{1}{\sqrt{5}}+\frac{1}{2}} (6x - 6x^2)dx$$

$$P\left[\left(-\frac{1}{\sqrt{5}} + \frac{1}{2}\right) < x < \left(\frac{1}{\sqrt{5}} + \frac{1}{2}\right)\right] = (3x^2 - 2x^3)\Big|_{-\frac{1}{\sqrt{5}}+\frac{1}{2}}^{\frac{1}{\sqrt{5}}+\frac{1}{2}} \cong 0.9838$$

7. $X \sim U(a, 10)$ dengan $\mu = 5$; $\sigma^2 = \frac{25}{3}$; $\sigma = \frac{5}{\sqrt{3}}$

Hitung $P[|x-5| > 4]$

$\mu = \frac{a+b}{2} \rightarrow 5 = \frac{a+10}{2} \rightarrow a = 0$

$r\sigma = 4$

$\frac{5}{\sqrt{3}} r = 4$

$r = \frac{4\sqrt{3}}{5}$

- Peluang kesamaan chebyshev

$P[|x-5| > 4] \leq \frac{1}{r^2}$

$P[|x-5| > 4] \leq \frac{1}{\left(\frac{4\sqrt{3}}{5}\right)^2}$

$P[|x-5| > 4] \leq \frac{25}{48}$

- $P[|x-5| > 4] = 1 - P[|x-5| < 4]$

$P[|x-5| > 4] = 1 - P(1 < x < 9)$

$P[|x-5| > 4] = 1 - \int_1^9 \frac{1}{10} dx$

$P[|x-5| > 4] = \left(1 - \frac{x}{10}\right)\Big|_1^9$

$P[|x-5| > 4] = 1 - \left(\frac{9-1}{10}\right)$

$P[|x-5| > 4] = \frac{1}{5}$

Bisa dilihat bahwa nilai peluang yang sebenarnya yaitu $\frac{1}{5}$ yang merupakan bagian/*subset* dari peluang kesamaan Chebyshev dimana $\frac{1}{5} \leq \frac{25}{48}$

8. Dik :

$X \sim U(-1, 3)$

$a = -1$

$b = 3$

$P[|x-\mu| \geq 2\sigma]$

$\mu = \frac{a+b}{2} = \frac{3+(-1)}{2} = 1$

$\sigma^2 = \frac{1}{12}(b-a)^2 = \frac{1}{12}(3-(-1))^2 = \frac{4}{3}$

$\sigma = \frac{2}{\sqrt{3}}$

$P[|x-\mu| \geq 2\sigma] \leq \frac{1}{r^2}$

$r = 2$

- Peluang Kesamaan Chebyshev

  $P\left[|x-\mu| \geq 2\left(\frac{2}{\sqrt{3}}\right)\right] \leq \frac{1}{2^2}$

  $P\left[|x-1| \geq 2\left(\frac{2}{\sqrt{3}}\right)\right] \leq \frac{1}{4}$

  $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] \leq \frac{1}{4}$

- $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] = 1 - P\left[|x-1| < \frac{4}{\sqrt{3}}\right]$

  $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] = 1 - P\left[\left(-\frac{4}{\sqrt{3}}+1\right) < x < \left(\frac{4}{\sqrt{3}}+1\right)\right]$

  $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] = 1 - \int_{-\frac{4}{\sqrt{3}}+1}^{\frac{4}{\sqrt{3}}+1} \frac{1}{4}dx$

  $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] = 1 - \left(\frac{1}{4}x\right)\Big|_{-\frac{4}{\sqrt{3}}+1}^{\frac{4}{\sqrt{3}}+1}$

  $P\left[|x-1| \geq \frac{4}{\sqrt{3}}\right] = 1 - \frac{2\sqrt{3}}{3} \cong 0{,}1547$

Bisa dilihat bahwa nilai peluang yang sebenarnya yaitu sekitar 0,1574 yang merupakan bagian/*subset* dari peluang kesamaan Chebyshev dimana $0{,}1574 \leq \frac{1}{4}$

9. Dik :

   $\mu = 20$ kg/hari

   $\sigma^2 = 16$

   $\sigma = 4$

   Ditanyakan

   a. P (X > 40)

   b. P (10 < X < 30)

   Jawab :

   a. $P[X > 40] \leq \frac{E(g(x))}{k}$

      $P[g(x) > 40] \leq \frac{20}{40}$

      $P[g(x) > 40] \leq \frac{1}{2}$

   b. $P(10 < x < 30) = P(\frac{10-20}{4} < Z < \frac{30-20}{4})$

      $P\left(\frac{10-20}{4} < Z < \frac{30-20}{4}\right) = P(-2{,}5 < Z < 2{,}5) = 0{,}9876$

      Atau dengan kesamaan Chebyshev

      $P[|x-20| < 10] \geq 1 - \frac{1}{r^2}$ dengan $10 = r\sigma = 4r \rightarrow r = \frac{5}{2}$

      $P[|x-20| < 10] \geq 1 - \frac{1}{\left(\frac{5}{2}\right)^2} \rightarrow P[|x-20| < 10] \geq \frac{21}{25}$

10. Dik :

$\mu = 120$

$\sigma^2 = 8$

$\sigma = 2\sqrt{2}$

Dit : a. $P[|x - \mu| \geq 810] \leq \frac{1}{r^2}$

b. $P[|x - \mu| \leq 120] \geq 1 - \frac{1}{r^2}$

Jawab:

a. $P[|x - \mu| \geq 810] \leq \frac{1}{r^2}$

$r\sigma = 810$

$2r\sqrt{2} = 810$

$r = \frac{405}{\sqrt{2}}$

$P[|x - 120| \geq 810] \leq \frac{1}{\left(\frac{405}{\sqrt{2}}\right)^2}$

$P[|x - 120| \geq 810] \leq \frac{2}{(405)^2}$

$P[|x - 120| \geq 810] \leq \frac{2}{164025}$

b. $P[|X - \mu| \leq 120] \geq 1 - \frac{1}{r^2}$

$r\sigma = 120$

$2r\sqrt{2} = 120$

$r = \frac{60}{\sqrt{2}}$

$P[|x - 120| \leq 120] \geq 1 - \frac{1}{\left(\frac{60}{\sqrt{2}}\right)^2}$

$P[|x - 120| \leq 120] \geq 1 - \frac{2}{(60)^2}$

$P[|x - 120| \leq 120] \geq 1 - \frac{1}{1800}$

$P[|x - 120| \leq 120] \geq \frac{1799}{1800}$

11. Dik :

$\mu = 8$

$\sigma^2 = 9$

$\sigma = 3$

Dit :

a. $P(-4 < x < 20)$

b. $P[|x - 8| \geq 6]$

Jawab :

Cara Kesamaan Chebyshev

a. $P[|x - 8| < 12] \geq 1 - \frac{1}{r^2}$

$r\sigma = 12$

$3r = 12$

$r = 4$

$P[|x - 8| < 12] \geq 1 - \frac{1}{4^2}$

$P[|x - 8| < 12] \geq \frac{15}{16}$

b. $P[|x-8| \geq 6] \leq \frac{1}{r^2}$

$r\sigma = 6$

$3r = 6$

$r = 2$

$P[|x-8| \geq 6] \leq \frac{1}{2^2}$

$P[|x-8| \geq 6] \leq \frac{1}{4}$

12. Y ~ B (n ; 0,25)

$P\left[\left|\frac{y}{n} - 0{,}25\right| < 0{,}05\right]$

a. Untuk n = 100

$$P\left[\left|\frac{y}{100} - 0{,}25\right| < 0{,}05\right] \geq 1 - \frac{1}{r^2}$$

$$P[|y - 25| < 5] \geq 1 - \frac{1}{r^2}$$

$\mu = n\,p$

$\mu = 100\,(0{,}25)$

$\mu = 25$

$\sigma^2 = n\,p\,q$

$\sigma^2 = 100\,(0{,}25)\,(0{,}75)$

$\sigma^2 = 18{,}75$

$$\sigma = \frac{5\sqrt{3}}{2}$$

$r\sigma = 5$

$$r = \frac{10}{5\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$P\left[\left|\frac{y}{100} - 0{,}25\right| < 0{,}05\right] \geq 1 - \frac{1}{\left(\frac{2}{\sqrt{3}}\right)^2}$$

$$P\left[\left|\frac{y}{100} - 0{,}25\right| < 0{,}05\right] \geq \frac{1}{4}$$

b. Untuk n = 500

$$P\left[\left|\frac{y}{500} - 0{,}25\right| < 0{,}05\right] \geq 1 - \frac{1}{r^2}$$

$$P[|y - 125| < 25] \geq 1 - \frac{1}{r^2}$$

$\mu = n\,p$

$\mu = 500\,(0{,}25)$

$\mu = 125$

$\sigma^2 = n\,p\,q$

$\sigma^2 = 500\,(0{,}25)\,(0{,}75)$

$\sigma^2 = 93{,}75$

$$\sigma = \frac{5\sqrt{15}}{2}$$

$r\sigma = 25$

$$\frac{5r\sqrt{15}}{2} = 25$$

$$r = \frac{10}{\sqrt{15}}$$

$$P\left[\,\left|\frac{y}{500} - 0{,}25\right| < 0{,}05\,\right] \geq 1 - \frac{1}{\left(\frac{10}{\sqrt{15}}\right)^2}$$

$$P\left[\,\left|\frac{y}{500} - 0{,}25\right| < 0{,}05\,\right] \geq 0{,}85$$

c. Untuk n = 1000

$$P\left[\,\left|\frac{y}{1000} - 0{,}25\right| < 0{,}05\,\right] \geq 1 - \frac{1}{r^2}$$

$$P\left[\,|y - 250| < 50\right] \geq 1 - \frac{1}{r^2}$$

$$\mu = n\,p$$
$$\mu = 1000\ (0{,}25)$$
$$\mu = 250$$

$$\sigma^2 = n\,p\,q$$
$$\sigma^2 = 1000\ (0{,}25)\ (0{,}75)$$
$$\sigma^2 = 187{,}5$$
$$\sigma = \frac{5\sqrt{30}}{2}$$

$$r\sigma = 50$$
$$r = \frac{20}{\sqrt{30}}$$

$$P\left[\,\left|\frac{y}{1000} - 0{,}25\right| < 0{,}05\,\right] \geq 1 - \frac{1}{r^2}$$

$$P\left[\,\left|\frac{y}{1000} - 0{,}25\right| < 0{,}05\,\right] \geq 1 - \frac{1}{\left(\frac{20}{\sqrt{30}}\right)^2}$$

$$P\left[\,\left|\frac{y}{1000} - 0{,}25\right| < 0{,}05\,\right] \geq 0{,}925$$

13. Diketahui

$\mu = \frac{7}{2}$

$\sigma^2 = \frac{5}{2}$

$k\sigma = \frac{5}{2}$

$P\ [\ |X - 3{,}5| < 2{,}5\ ]$

Ditanyakan : Tentukan k dan batas bawah dari $P\ [\ |X - 3{,}5| < 2{,}5\ ]$

Jawab :

$\sigma^2 = \frac{5}{2} \rightarrow \sigma = \sqrt{\frac{5}{2}}$

$k\sigma = \frac{5}{2} \rightarrow \text{k} = \frac{\frac{5}{2}}{\sqrt{\frac{5}{2}}} = \sqrt{\frac{5}{2}}$

Batas bawah

$P\ [\ |X - 3{,}5| < 2{,}5] \geq 1 - \frac{1}{k^2}$

$P\ [\ |X - 3{,}5| < 2{,}5] \geq 1 - \frac{1}{\frac{5}{2}}$

$P\ [\ |X - 3{,}5| < 2{,}5] \geq 1 - \frac{2}{5}$

$P\ (\ |X - 3{,}5| < 2{,}5\ ) \geq \frac{3}{5}$

TERIMA KASIH